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Comparison of Receiver Operating Characteristics of Four Different Processors for the Detection of Random Signal Transmissions

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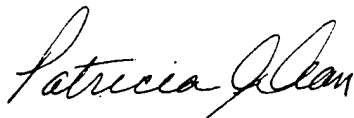
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PREFACE

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13. ABSTRACT (Maximum 200 words) <p>When a segment of a random sequence is used as the transmitted signal waveform and is made available to a receiver attempting to detect the presence of a weak target echo, an extra degree of freedom is injected into the optimum processing scheme. Optimum and near-optimum processing of the received data then involves neither the energy detector nor the standard crosscorrelator; rather, a normalized crosscorrelator is found to perform virtually at the level of the optimum processor.</p> <p>The characteristic functions of the decision variables of all four processors (namely, the energy detector, crosscorrelator, normalized crosscorrelator, and optimum processor) are derived in closed form in terms of M, the signal time-bandwidth product, and R, the input signal-to-noise ratio per sample. These analytic results are numerically transformed to yield the corresponding receiver operating characteristics for a wide range of values of false alarm and detection probabilities.</p>				
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LIST OF ABBREVIATIONS, ACRONYMS, AND SYMBOLS

a	Medium scaling or attenuation
C	Crosscorrelator output
CC	Crosscorrelator
CF	Characteristic function
d^2	Deflection criterion
E	Received signal energy
E_C	Exceedance distribution function of random variable C
ED	Energy detector
EDF	Exceedance distribution function
E_T	Transmitted signal energy
FFT	Fast Fourier transform
f_w	Characteristic function of scaled variable w
f_z	Characteristic function of decision variable z
GLR	Generalized likelihood ratio
H_0, H_1	Signal-absent, signal-present hypotheses
LR	Likelihood ratio
m	Time index
M	Total number of samples; time-bandwidth product; signal extent
m_a	Mean of Gaussian random variable a
M_b	Shift value
M_f	Size of FFT
ML	Maximum likelihood
N	$M/2$ (integer)
n_m	Noise value at time index m
NC	Normalized crosscorrelator
OP	Optimum processor
p_a	Probability density function of scaling a
p_C	Probability density function of C
P_d	Detection probability
PDF	Probability density function
P_f	False alarm probability
p_Q	Probability density function of sample energy Q
$p(X)$	Probability density function of observation X
Q	Sample transmitted signal energy
R	Received signal-to-noise ratio
r_m	Normalized reference value at time index m
ROC	Receiver operating characteristic
RV	Random variable
s_m	Signal value at time index m
T	Fixed threshold
u	Argument of probability density functions
w	Scaled decision variable
X	Data or observation $\{x_1, \dots, x_M\}$

LIST OF ABBREVIATIONS, ACRONYMS, AND SYMBOLS (Cont'd)

x_m	Received data sample at time index m
z	Decision variable
α	Switch parameter: 0 for H_0 , 1 for H_1
β	Switch parameter: -1 for H_0 , 1 for H_1
μ_k	Mean under hypothesis k
ξ	Argument of characteristic functions
σ_a	Standard deviation of Gaussian random variable a
σ_n	Standard deviation of noise sequence $\{n_m\}$
σ_r	Standard deviation of received signal $\{as_m\}$
σ_s	Standard deviation of signal sequence $\{s_m\}$
Φ	Distribution function for normalized Gaussian variable
overbar	Ensemble average

COMPARISON OF RECEIVER OPERATING CHARACTERISTICS OF FOUR DIFFERENT PROCESSORS FOR THE DETECTION OF RANDOM SIGNAL TRANSMISSIONS

INTRODUCTION

In an effort to avoid interception, a random transmitted signal is sometimes used for detection and communication purposes. It is presumed here, for the most part, that the receiver (which may or may not be co-located with the transmitter) has precise knowledge of the particular transmitted random signal sequence, but not of the medium attenuation. The receiver's goal is to detect any target echo, if present, with a maximum detection probability, while maintaining a small, fixed false alarm probability.

Various processors are candidates for signal detection in this scenario, among which are

- the energy detector,
- the crosscorrelator,
- the normalized crosscorrelator, and
- the optimum detector.

The last processor and its detectability performance serve as a standard of comparison for the first three practical processors. Comparison of these four processors is desired for time-bandwidth products ranging from small values to large values and for all ranges of received signal-to-noise ratio.

Under the conditions to be indicated explicitly below, the characteristic functions of the decision variables for all four processors have been derived in closed form. A numerical procedure utilizing a fast Fourier transform (FFT) then yields the entire exceedance distribution function, from which the receiver operating characteristics are readily found. This exact approach

avoids the need to resort to approximations such as the deflection criterion, with its limited applicability. The misleading character of the deflection for the four processors of interest will be discussed later in the report.

OPTIMUM PROCESSORS FOR VARIOUS SIGNAL MODELS

The transmitted signal sequence is characterized by $\{s_m\}$ for $1 \leq m \leq M$, where M is the number of signal samples. This sequence, whether deterministic or random, is known to the receiver. The received or measured data sequence, under the signal-absent hypothesis H_0 or the signal-present hypothesis H_1 , respectively, is

$$x_m = \begin{cases} n_m & \text{for } H_0 \\ as_m + n_m & \text{for } H_1 \end{cases} \quad \text{for } 1 \leq m \leq M,$$

where scaling a is the medium attenuation. This scaling a is constant over the signal extent M ; however, the knowledge of a or its statistics may vary with the situation. The additive noise $\{n_m\}$ is stationary white Gaussian with zero mean and known variance σ_n^2 . M can also be interpreted as the time-bandwidth product of the signal.

The optimum and near-optimum processors will be derived under a variety of assumptions regarding the character or knowledge of scaling a . For comparison, the situation of a deterministic transmitted signal sequence is addressed first; then, the alternative current situation of a random transmitted signal is undertaken. The derivations of the performances of the resultant processors, in terms of false alarm and detection probabilities, are considered in the next section.

DETERMINISTIC TRANSMITTED SIGNAL

In this subsection, not only is transmitted sequence $\{s_m\}$ known to the receiver, but it has *no variability* to it. Thus, its sample (transmitted) energy

$$Q \equiv \sum_{m=1}^M s_m^2$$

is a known *constant*. For a given fixed value of medium attenuation a , the probability density

functions (PDFs) of observation $X = \{x_1, \dots, x_M\}$ under H_0 and H_1 are

$$p_0(X) = \prod_{m=1}^M \left[(2\pi\sigma_n^2)^{-1/2} \exp\left(-\frac{x_m^2}{2\sigma_n^2}\right) \right]$$

and

(1)

$$p_1(X) = \prod_{m=1}^M \left[(2\pi\sigma_n^2)^{-1/2} \exp\left(-\frac{(x_m - as_m)^2}{2\sigma_n^2}\right) \right],$$

using the knowledge of $\{s_m\}$. For this fixed value of a , the likelihood ratio (LR) is

$$\text{LR}(X) = \frac{p_1(X)}{p_0(X)} = \exp\left(\frac{a}{\sigma_n^2} C - \frac{a^2}{2\sigma_n^2} Q\right),$$

where

$$C \equiv \sum_{m=1}^M s_m x_m$$

is the crosscorrelation of the known signal $\{s_m\}$ with the received data $\{x_m\}$. Several cases of knowledge of scaling a will now be considered.

Case I: $a = \text{Known Constant}$

In this case, the quantity $\frac{1}{2}a^2 Q/\sigma_n^2$ is a known constant. Using the monotonicity of the $\exp(\cdot)$ function, the LR test is

$$C \underset{<}{>} T,$$

where fixed threshold T absorbs all the known constants α , Q , and σ_n .

Case II: $\alpha = \text{Unknown Constant}$

Now, $\frac{1}{2}\alpha^2 Q/\sigma_n^2$ is unknown, but it is *constant*; that is, it does not vary from trial to trial.

Therefore, the LR test is again

$$C \underset{<}{>} T,$$

where fixed threshold T absorbs the unknown constant α and the known constants Q and σ_n .

Case III: $\alpha = \text{Random Variable}$

If random variable (RV) α has known PDF p_a , the PDF $p_1(X)$ in equation (1) must be averaged over variable α . The modified LR is now

$$\text{LR}(X) = \frac{\int du p_a(u) p_1(X|\alpha = u)}{p_0(X)} = \int du p_a(u) \exp\left(\frac{u}{\sigma_n^2} C - \frac{u^2}{2\sigma_n^2} Q\right). \quad (2)$$

If random variable α is limited to positive values, this LR is monotonic with quantity C , which may be seen from

$$\frac{\partial}{\partial C} \text{LR}(X) = \int_0^\infty du p_a(u) \frac{u}{\sigma_n^2} \exp\left(\frac{u}{\sigma_n^2} C - \frac{u^2}{2\sigma_n^2} Q\right).$$

Because the integrand is everywhere positive, it follows that the right-hand side is positive for all C , regardless of *known* constant Q ; that is,

$$\text{LR}(X) = L(C; Q, \sigma_n^2, p_a)$$

in equation (2) is a monotonically increasing function of C . Since Q , σ_n^2 , and the parameters of PDF p_a are constant from trial to trial, the LR test is simply

$$C \underset{<}{>} T,$$

where fixed threshold T includes Q , σ_n^2 , and the parameters of p_a .

Notice that the LR test here is always a comparison of crosscorrelator output C with a *fixed* threshold, *regardless of the PDF* p_a of positive amplitude-scaling a . If a could take on negative values, this same conclusion need not hold true. Thus, for a *deterministic* signal $\{s_m\}$, the optimum processor is always $C \underset{<}{>} T$, regardless of the knowledge or statistics of scaling a (if positive). This is the standard comparison of the crosscorrelator output with a fixed threshold.

RANDOM TRANSMITTED SIGNAL

Here, signal segment $\{s_m\}$ will be drawn from a random sequence, which need not be stationary or white. The particular values $\{s_m\}$ are again *known* to the receiver, but will *vary* from trial to trial. Now, the signal sample (transmitted) energy

$$Q = \sum_{m=1}^M s_m^2$$

is known to the receiver, but it *varies* from trial to trial. For a given fixed value of attenuation a , the LR is again

$$\text{LR}(X) = \exp \left[\frac{a}{\sigma_n^2} C - \frac{a^2}{2\sigma_n^2} Q \right], \quad (3)$$

using the *knowledge* of the particular samples $\{s_m\}$. Again, three cases of knowledge about scaling a are of interest.

Case I: $a = \text{Known Constant}$

In this case, the quantity $\frac{1}{2}a^2 Q/\sigma_n^2$ is known at the receiver, but it *varies* from trial to trial. Therefore, the LR test can only be simplified to

$$C - \frac{1}{2}aQ \underset{<}{>} T,$$

where T is a *fixed* threshold. The quantity $\frac{1}{2}aQ$ *must* be subtracted from C because it varies with each trial; only after this subtraction is a comparison with a *fixed* threshold valid as the LR test. The test $C \underset{>}{<} T$ is no longer the optimum processor when $\{s_m\}$ is a segment of a random process, even though the entire sequence $\{s_m\}$ is known to the receiver.

An alternative way to express the LR test is

$$\sum_{m=1}^M x_m^2 - \sum_{m=1}^M (x_m - as_m)^2 \underset{<}{>} T.$$

This expression is a *difference* of two energy estimates: one with the data $\{x_m\}$ as is, the other with the received signal subtracted. The leading term (by itself as a processor) is called the energy detector.

If the number of samples, M , in the signal segment is large, the quantity Q , although random, will not vary much from trial to trial. In this special case, the term $\frac{1}{2}aQ$ will be substantially constant, and the LR test becomes approximately $C \underset{>}{<} T$ for large M , where *fixed* threshold T absorbs the almost-fixed term $\frac{1}{2}aQ$.

Observe that the statistics of signal $\{s_m\}$ were never needed for the above derivations of the optimum processor; these results apply for any segment of any random sequence $\{s_m\}$ known to the receiver. However, the *performance* of the LR test will need specification of the signal statistics.

Case II: $a = \text{Unknown Constant}$

Here the quantity $\frac{1}{2}aQ$ is unknown and varies from trial to trial. Therefore, the calculation of $C - \frac{1}{2}aQ$ cannot be carried out, and the optimum processor cannot be realized, because a is unknown.

When the signal samples $\{s_m\}$ for $1 \leq m \leq M$ happen to have a high level, then Q is also high level and the subtracted term tends to compensate for that effect on C . Similarly, a low-level signal sample set is subject to the same type of compensation. That is, the optimum processor is compensating for the effects of a variable signal transmission level. This suggests the following suboptimum processor that also compensates, but does *not* need knowledge of unknown gain a , namely,

$$\frac{C}{\sqrt{Q}} \underset{<}{\overset{>}{T}},$$

where T is a *fixed* threshold. The alternative equivalent expression,

$$\sum_{m=1}^M r_m x_m \underset{<}{\overset{>}{T}}, \quad \text{with } r_m = \frac{s_m}{\sqrt{Q}} \quad \text{for } 1 \leq m \leq M,$$

reveals that the local reference $\{r_m\}$ is independent of the absolute level of the particular signal segment, $\{s_m\}$, employed.

Another approach, in this case of an unknown parameter, uses the method of maximum likelihood (ML). The a -dependency of PDF $p_1(X)$ is given by equation (3); therefore, the ML estimate is $\hat{a} = 0$ for $C < 0$ and

$$\hat{a} = \frac{C}{Q} = \frac{\sum_{m=1}^M s_m x_m}{\sum_{m=1}^M s_m^2} \quad \text{for } C \geq 0.$$

This choice reflects the fact that a is known to be positive. The corresponding generalized likelihood ratio (GLR) is

$$\text{GLR}(X) = \begin{cases} \exp\left(\frac{1}{2\sigma_n^2} \frac{C^2}{Q}\right) & \text{for } C \geq 0 \\ 1 & \text{for } C < 0 \end{cases},$$

which is monotonic in C . The GLR leads to the test

$$\frac{C}{\sqrt{Q}} \underset{<}{\overset{>}{T}} \quad (\text{where } T > 1),$$

in agreement with the compensation argument suggested above. This processor is called the normalized crosscorrelator or normalized matched filter for random signal transmissions.

It should be observed again that knowledge of the signal statistics was not needed or used in arriving at suboptimum processor C/\sqrt{Q} . Also, the earlier comment relative to a large sample-size M is relevant here as well; namely, Q is relatively constant then, and the LR test is well approximated by

$$C \underset{<}{\overset{>}{T}} \quad \text{for large } M,$$

where T is a fixed threshold.

Case III: a = Random Variable

The LR is again given by equation (2); that is,

$$\text{LR}(X) = \int du p_a(u) \exp\left(\frac{u}{\sigma_n^2} C - \frac{u^2}{2\sigma_n^2} Q\right).$$

But now, both C and Q vary with each trial; it is no longer sufficient to consider $\partial(\text{LR})/\partial C$.

In order to proceed further in this case, it is necessary to specify the particular PDF p_a .
Let us consider

$$p_a(u) = (2\pi\sigma_a^2)^{-1/2} \exp\left[-\frac{(u - m_a)^2}{2\sigma_a^2}\right],$$

where mean m_a and variance σ_a^2 of Gaussian RV a are presumed known. (If $m_a > 3\sigma_a$, the RV a is almost always positive, which is the situation of interest.) The LR then becomes

$$\text{LR}(X) = \frac{\sigma_n}{(\sigma_n^2 + \sigma_a^2 Q)^{1/2}} \exp\left[-\frac{m_a^2}{2\sigma_a^2} + \frac{(\sigma_n^2 m_a + \sigma_a^2 C)^2}{2\sigma_a^2 \sigma_n^2 (\sigma_n^2 + \sigma_a^2 Q)}\right].$$

The quantity C is more variable than Q from trial to trial. Also, the exponential accentuates variations far more than the square root in the denominator. Therefore, the LR test is well approximated by

$$\frac{(\sigma_n^2 m_a + \sigma_a^2 C)^2}{\sigma_n^2 + \sigma_a^2 Q} > T,$$

where T is a fixed threshold. Furthermore, $\sigma_n^2 m_a + \sigma_a^2 C$ very seldom becomes negative, allowing a modification to the approximate test

$$\frac{\sigma_n^2 m_a + \sigma_a^2 C}{\sqrt{\sigma_n^2 + \sigma_a^2 Q}} > T.$$

If all the parameters m_a , σ_a , and σ_n are known, this test should yield near-optimum detection. However, if there is some uncertainty, or if the amplitude PDF p_a is not really Gaussian, the major dependency of this test is preserved by the approximation

$$\frac{C}{\sqrt{Q}} \underset{<}{>} T,$$

where T is a fixed threshold. Alternatively, this test can be written as

$$\sum_{m=1}^M r_m x_m \underset{<}{>} T, \quad r_m = \frac{s_m}{\sqrt{Q}} \quad \text{for } 1 \leq m \leq M,$$

giving a local reference $\{r_m\}$ with no random variations.

In summary, for a random signal, a near-optimum processor is the normalized crosscorrelator or normalized matched filter

$$\sum_{m=1}^M r_m x_m = \frac{\sum_{m=1}^M s_m x_m}{\left(\sum_{m=1}^M s_m^2 \right)^{1/2}} \underset{<}{>} T,$$

regardless of the knowledge or statistics of scaling a (if positive). The only exception is when a is a known constant, for which the LR test is $C - \frac{1}{2}aQ \underset{<}{>} T$; however, this is not the typical practical application.

The signal statistics are irrelevant to all these processor developments above, although they will affect the performances. This occurs because the particular signal samples, $\{s_m\}$ for $1 \leq m \leq M$, are made available to the receiver attempting to decide on signal presence or absence in that particular measurement.

PERFORMANCE

In general, the received waveform sequence is

$$x_m = \alpha a s_m + n_m, \quad \alpha = \begin{cases} 0 & \text{for } H_0 \\ 1 & \text{for } H_1 \end{cases}.$$

The average received noise power is $\sigma_n^2 = \overline{n_m^2}$ (where the overbar represents an ensemble average), while the received signal energy (when present) is defined as

$$E = \sum_{m=1}^M (a s_m)^2 = a^2 Q.$$

DETERMINISTIC TRANSMITTED SIGNAL

The only processor of interest in this subsection is the crosscorrelator (matched filter) with output

$$C = \sum_{m=1}^M s_m x_m = \sum_{m=1}^M s_m (\alpha a s_m + n_m) = \alpha a Q + \sum_{m=1}^M s_m n_m.$$

For a given value of α , C is a Gaussian random variable with mean $\alpha a Q$ and variance $Q \sigma_n^2$.

Therefore, the PDF of C is

$$p_C(u) = (2\pi Q \sigma_n^2)^{-1/2} \exp \left[-\frac{(u - \alpha a Q)^2}{2Q \sigma_n^2} \right],$$

and the exceedance distribution function (EDF) of C is

$$E_C(v) = \int_v^\infty du p_C(u) = \Phi \left(\alpha \frac{\sqrt{E}}{\sigma_n} - T \right), \quad T \equiv \frac{v}{\sqrt{Q} \sigma_n},$$

where the distribution function for the normalized Gaussian variable is

$$\Phi(x) \equiv \int_{-\infty}^x dt (2\pi)^{-1/2} \exp\left(-\frac{t^2}{2}\right).$$

If α is set at 0 and then α is set at 1, the false alarm and detection probabilities follow as

$$P_f = \Phi(-T) \quad \text{and} \quad P_d = \Phi\left(\frac{\sqrt{E}}{\sigma_n} - T\right),$$

respectively, for a given value of signal energy $E = a^2 Q$.

If the scaling factor a is a random variable, then P_d must be averaged over a , using PDF p_a .

For example, if a is Gaussian with mean m_a and variance σ_a^2 , then the average P_d is

$$\bar{P}_d = \Phi\left(\frac{\frac{m_a}{\sigma_n} \sqrt{Q} - T}{\sqrt{1 + \frac{\sigma_a^2}{\sigma_n^2} Q}}\right).$$

(If $\sigma_a = 0$, this reduces to P_d above, because $\sqrt{E} = a\sqrt{Q} = m_a\sqrt{Q}$.)

RANDOM TRANSMITTED SIGNAL

Now, once again, there are several processors of interest:

- Optimum processor $C - \frac{1}{2}aQ \gtrless T$ (known a),
- Normalized crosscorrelator $\frac{C}{\sqrt{Q}} \gtrless T$,

- Crosscorrelator $C \begin{matrix} > \\ < \end{matrix} T$, and
- Energy detector $\sum_{m=1}^M x_m^2 \begin{matrix} > \\ < \end{matrix} T$.

In order to make progress, it is necessary to assign some statistics to signal segment $\{s_m\}_1^M$. It is presumed that the signal sequence is stationary white Gaussian, with zero mean and variance σ_s^2 .

For a given value of attenuation a , the characteristic functions (CFs) of the various decision variables will be derived in closed form. Since the average received signal power is

$$\sigma_r^2 \equiv \overline{(as_m)^2} = a^2 \sigma_s^2,$$

the results can be equally well conditioned on σ_r^2 or the dimensionless quantity

$$R = \frac{\sigma_r^2}{\sigma_n^2} = \frac{a^2 \sigma_s^2}{\sigma_n^2},$$

which is the received or input signal-to-noise power ratio per sample. Thus, all the following results will be conditioned on a given value of R . If attenuation a is random and its PDF is known, these results can be averaged over the statistics of attenuation a to find the average detection probability.

The EDFs for all the above processors cannot be found in any convenient form. Therefore, it is necessary to resort to the accurate and efficient FFT procedure described in the reference* for proceeding directly from the CF to the EDF. For example, false alarm probabilities in the E -10 range are quickly available and accurate.

* Albert H. Nuttall, "Accurate Efficient Evaluation of Cumulative or Exceedance Probability Distributions Directly from Characteristic Functions," NUSC Technical Report 7023, Naval Underwater Systems Center, New London, CT, 1 October 1983.

Optimum Processor for Known a

The decision variable for this processor is given by

$$z \equiv C - \frac{1}{2}aQ = \sum_{m=1}^M s_m (\alpha a s_m + n_m) - \frac{1}{2}aQ = (\alpha - \frac{1}{2})aQ + \sum_{m=1}^M s_m n_m.$$

If the signal sample values $\{s_m\}$ are held fixed for the moment, the conditional CF of z is

$$f_z(\xi | \{s_m\}) = \overline{\exp(i\xi z)} = \exp \left[i\xi (\alpha - \frac{1}{2})aQ - \frac{1}{2} \xi^2 \sigma_n^2 Q \right],$$

which is based on the fact that $\{n_m\}$ is a zero-mean white Gaussian sequence. Then, averaging over the Gaussian statistics of $\{s_m\}$ results in the CF of z as

$$f_z(\xi) = \left[1 + \xi^2 \sigma_s^2 \sigma_n^2 - i\xi(2\alpha - 1)a\sigma_s^2 \right]^{-M/2}.$$

Since the receiver operating characteristics (ROCs) are not affected by a scale change, consider the scaled decision variable $w = z/(\sigma_s \sigma_n)$, for which the CF is

$$f_w(\xi) = \left(1 + \xi^2 - i\xi\beta\sqrt{R} \right)^{-M/2}, \quad \beta = 2\alpha - 1 = \begin{cases} -1 & \text{for } H_0 \\ 1 & \text{for } H_1 \end{cases}.$$

For a given value of R , EDF $E_0(v)$ under H_0 is found from f_w by setting switch parameter β to -1, while EDF $E_1(v)$ is available by setting β to 1. Then, by eliminating threshold v , the ROC, P_d versus P_f , can be plotted for that particular value of R . This procedure must be repeated for each R value of interest. (For $R = 0$, the two CFs are equal, leading to $P_d = P_f$, as expected.)

An aid in the numerical evaluation of the EDFs discussed here is to note that

$$f_0(\xi) = f_1(-\xi) , \quad p_0(u) = p_1(-u) ,$$

leading to

$$E_0(v) \int_v^{\infty} du p_0(u) = 1 - E_1(-v) .$$

Thus, only $E_1(v)$ need be numerically evaluated.

Normalized Crosscorrelator

Now, the decision variable is

$$z \equiv \frac{C}{\sqrt{Q}} = \frac{1}{\sqrt{Q}} \sum_{m=1}^M s_m (\alpha a s_m + n_m) = \alpha a \sqrt{Q} + \frac{1}{\sqrt{Q}} \sum_{m=1}^M s_m n_m .$$

For any given sequence of signal samples $\{s_m\}$ for $1 \leq m \leq M$, the last term in z is a Gaussian RV with zero mean and variance σ_n^2 ; thus, the conditional CF of z is

$$f_z(\xi | \{s_m\}) = \exp \left(i \xi \alpha a \sqrt{Q} - \frac{1}{2} \xi^2 \sigma_n^2 \right) .$$

And since Q is a sum of M squared Gaussian RVs, its PDF is

$$p_Q(u) = \frac{u^{N-1} \exp(-u/2\sigma_s^2)}{\Gamma(N)(2\sigma_s^2)^N} \quad \text{for } u > 0, \quad N = \frac{M}{2} .$$

This leads to the unconditional CF of z as

$$f_z(\xi) = \exp\left(-\frac{1}{2} \xi^2 \sigma_n^2\right) \int_0^{\infty} du \exp(i \xi \alpha a \sqrt{u}) \frac{u^{N-1} \exp(-u/2\sigma_s^2)}{\Gamma(N)(2\sigma_s^2)^N} .$$

For scaled decision variable $w = z/\sigma_n$, and with the change of variable $u = \sigma_s^2 t^2$, the pertinent CF is

$$f_w(\xi) = \exp\left(-\frac{1}{2}\xi^2\right) \int_0^\infty dt \exp(i\xi\alpha\sqrt{R}t) \frac{t^{2N-1} \exp(-t^2/2)}{\Gamma(N)2^{N-1}}.$$

Although this result is not in closed form, it can easily and efficiently be numerically evaluated for any given R by means of a single FFT. Under H_0 , this reduces to $f_w(\xi|H_0) = \exp(-\xi^2/2)$ for arbitrary M . The alternative special case of $M = 2$ can be expressed as

$$f_w(\xi|H_1) = \exp\left(-\frac{1}{2}\xi^2\right) \left[1 + i\xi\left(\frac{\pi}{2}R\right)^{1/2} w\left(\xi\left(\frac{R}{2}\right)^{1/2}\right) \right].$$

Then $P_f = \Phi(-T)$, where T is the fixed threshold against which w is compared.

Crosscorrelator

This processor has decision variable

$$z = C = \sum_{m=1}^M s_m (\alpha a s_m + n_m) = \alpha a Q + \sum_{m=1}^M s_m n_m.$$

The conditional CF of z is

$$f_z(\xi|\{s_m\}) = \exp\left(i\xi\alpha a Q - \frac{1}{2}\xi^2\sigma_n^2 Q\right).$$

Thus, using the Gaussian statistics of $\{s_m\}$, the unconditional CF of z is

$$f_z(\xi) = \left(1 + \xi^2\sigma_s^2\sigma_n^2 - i\xi 2\alpha a\sigma_s^2\right)^{-M/2}.$$

With scaled decision variable $w = z/(\sigma_s \sigma_n)$, the pertinent CF is

$$f_w(\xi) = \left(1 + \xi^2 - i\xi 2\alpha\sqrt{R}\right)^{-M/2}.$$

As usual, $\alpha = 0$ for H_0 and $\alpha = 1$ for H_1 .

Energy Detector

Finally, the last detector of interest has decision variable

$$z = \sum_{m=1}^M x_m^2 = \sum_{m=1}^M (\alpha a s_m + n_m)^2.$$

The RV x_m is zero-mean Gaussian with variance $\alpha a^2 \sigma_s^2 + \sigma_n^2$. Using the independence of the $\{x_m\}$ yields the CF of z as

$$f_z(\xi) = \left[1 - i\xi 2(\alpha a^2 \sigma_s^2 + \sigma_n^2)\right]^{-M/2}.$$

For scaled decision variable $w = z/\sigma_n^2$, there follows CF

$$f_w(\xi) = \left[1 - i\xi 2(1 + \alpha R)\right]^{-M/2}.$$

The quantity α is equal to 0 for H_0 and is equal to 1 for H_1 .

For M even and $N = M/2$, there follow probabilities

$$P_f = \exp\left(-\frac{T}{2}\right) \sum_{k=0}^{N-1} \frac{1}{k!} \left(\frac{T}{2}\right)^k$$

and

$$P_d = \exp\left(\frac{-T}{2(1+R)}\right) \sum_{k=0}^{N-1} \frac{1}{k!} \left(\frac{T}{2(1+R)}\right)^k,$$

where T is the threshold against which decision variable w is compared.

COMPARISON OF DEFLECTIONS

For large numbers of samples, M , it can be expected that the decision variables would tend to Gaussian. Then, if $P_d = 0.5$ were the desired level of detectability, the deflection

$$d^2 = \frac{(\mu_1 - \mu_0)^2}{\sigma_0^2}$$

is an important statistic in that it tends to govern P_f , at least for moderate values of P_f . Here, μ_k is the mean of the decision variable under hypothesis H_k , while σ_0 is the standard deviation under H_0 . However, since d only involves first and second moments of the decision variable, it can yield misleading results about which processors are best in terms of their ROCs; the ROCs involve *all* orders of moments and are therefore reliable measures for comparing processors.

For the optimum processor, the moments can be extracted from CF $f_w(\xi)$ as

$$\mu = \frac{M}{2} \beta \sqrt{R}, \quad \sigma^2 = M\left(1 + \frac{1}{2}R\right), \quad d^2 = M \frac{R}{1 + \frac{1}{2}R}.$$

For the normalized crosscorrelator, $N = M/2$ and

$$\mu = \alpha \sqrt{2R} \frac{\Gamma(N + \frac{1}{2})}{\Gamma(N)}, \quad \sigma_0^2 = 1, \quad d^2 = 2R \left[\frac{\Gamma(\frac{M+1}{2})}{\Gamma(\frac{M}{2})} \right]^2.$$

For large M , there follows $d^2 \approx MR$.

For the crosscorrelator,

$$\mu = \alpha M \sqrt{R}, \quad \sigma^2 = M(1 + \alpha 2R), \quad d^2 = MR.$$

Finally, for the energy detector,

$$\mu = M(1 + \alpha R), \quad \sigma^2 = M2(1 + \alpha R)^2, \quad d^2 = M \frac{1}{2} R^2.$$

The deflection for the energy detector indicates a small-signal suppression according to R^2 ; thus, for large M , where small R is of interest, the energy detector will perform very poorly relative to the other processors. This observation is borne out by numerical evaluation of the ROCs of all the processors.

On the other hand, for large R , the energy detector has the largest deflection, which grows as R^2 . Yet, the energy detector is not the best processor for large R , as the ROCs show. This is an example of the shortcomings of the deflection as a criterion for comparing processors.

The deflection criteria for the other three processors behave as MR for small R , indicating comparable performance. However, for large R , the deflection of the optimum processor *saturates* at $2M$, while the deflections for the two crosscorrelators increase as MR . Since the optimum processor outperforms both crosscorrelators, the inadequacy of the deflection is again illustrated.

In summary, the deflection criterion can be a very misleading statistic. It should be used only when higher order moments are absolutely unavailable, when the number of samples is large, when $P_d \approx 0.5$ is of interest, and when P_f is moderate (on the order of $E-2$ or $E-4$). In general, comparison of the ROCs is the only valid procedure.

NORMALIZATION BEFORE TRANSMISSION

When the random signal segment $\{s_m\}$ for $1 \leq m \leq M$ is drawn, suppose that it is normalized prior to transmission, according to

$$\tilde{s}_m \equiv \sqrt{E_T} \frac{s_m}{\sqrt{Q}} \quad \text{for } 1 \leq m \leq M ,$$

where E_T is the transmitted signal energy. Since the receiver knows $\{s_m\}$, it has no uncertainty about the transmitted signal, and this case reverts to the deterministic signal, even though the particular transmitted sequence varies from trial to trial. That is, the transmitted deterministic signal varies from trial to trial.

The received signal waveform under H_1 is $a\tilde{s}_m$, where a is the medium attenuation. The LR test is, as earlier,

$$C \underset{<}{>} T ,$$

and the performance is again

$$P_f = \Phi(-T) , \quad P_d = \Phi\left(\frac{\sqrt{E}}{\sigma_n} - T\right)$$

for a given value of the received signal energy

$$E = \sum_{m=1}^M (a\tilde{s}_m)^2 = a^2 E_T .$$

GRAPHICAL RESULTS

The ROCs for detection of a deterministic signal are given in figure 1. Because the received signal is not random in this case, these curves are parameterized in terms of the total received signal energy E . In particular, a decibel (dB) measure relative to the average received noise power is utilized. These characteristics are parallel straight lines for all decibel values.

All the remaining ROCs are for random transmitted signals and are parameterized with respect to the received average power signal-to-noise ratio R , namely, $R(\text{dB}) = 10 \log_{10}(R)$. The situation of a low number of samples, $M = 10$, is depicted in figure 2 for the optimum processor. (The quantity Δ_{ξ} is the increment in ξ used in computing the EDF directly from the corresponding CF; M_b and b are shift values guaranteeing a positive random variable; and M_f is the size of the FFT employed. These computational parameter values are presented in case the user desires to confirm or extend these results.) The corresponding results for the optimum processor with $M = 100$ and $M = 1000$ are displayed in figures 3 and 4, respectively, along with the required values of the computational parameters that guarantee accurate false alarm and detection probabilities.

Similar ROCs are presented in figures 5, 6, and 7 for the normalized crosscorrelator with $M = 10$, 100, and 1000, respectively. The corresponding results for the (unnormalized) crosscorrelator are given in figures 8, 9, and 10, while the energy detector results are depicted in figures 11, 12, and 13.

A low-quality operating point is defined as $P_f = 1E-3$ and $P_d = 0.5$, while a high-quality operating point is defined as $P_f = 1E-6$ and $P_d = 0.9$. The first observation to make is that the two "usual" processors, the energy detector (ED) and the crosscorrelator (CC), cannot be unequivocally ordered in terms of performance. For example, for the small time-bandwidth product of $M = 10$, reference to figures 8 and 11 shows that the CC outperforms the ED at the low-quality point, requiring a signal-to-noise ratio per sample of 1.6 dB versus 3.3 dB. On the other hand, at the high-quality point for $M = 10$, the ED is better than the CC, requiring only

9.3 dB versus 12.3 dB. However, for larger numbers of samples M , such as $M = 100$ in figures 9 and 12, or $M = 1000$ in figures 10 and 13, the CC considerably outperforms the ED for all signal-to-noise ratios.

Tables 1 and 2 below list the values of the input signal-to-noise ratio per sample (in decibels) required in order to achieve the low-quality and high-quality performance levels for the ED and CC, as well as for the normalized crosscorrelator (NC) and the optimum processor (OP). It can be seen from these results (or figures 2, 3, and 4 relative to figures 5, 6, and 7, respectively) that the NC performs virtually at the optimum performance level attainable by the OP. The only discrepancy is 0.2 dB for $M = 10$ at the high-quality operating point. This is particularly striking when it is recalled that the NC does not need any additional information about the medium attenuation, while the OP does need and use this information.

The rightmost column in the tables is derived from the deterministic signal results in figure 1. It does not constitute a strictly legal comparison case; it has been added only for completeness and curiosity. Furthermore, the values tabulated have been converted to a per-sample value by subtracting $10 \log_{10}(M)$ from the values obtained in figure 1, which measure the *total* received signal energy E . Nevertheless, the values in this last column are only slightly smaller than those for the OP and the NC, indicating rather close performance levels for the deterministic versus random signal cases.

Table 1. Required Signal-to-Noise Ratio for $P_f = 1E-3$ and $P_d = 0.5$

M	ED	CC	NC	OP	Deterministic
10	3.3	1.6	0.0	0.0	-0.2
100	-3.0	-10.0	-10.2	-10.2	-10.2
1000	-8.4	-20.2	-20.2	-20.2	-20.2

Table 2. Required Signal-to-Noise Ratio for $P_f = 1E-6$ and $P_d = 0.9$

M	ED	CC	NC	OP	Deterministic
10	9.3	12.3	7.3	7.1	5.6
100	0.8	-3.5	-4.2	-4.2	-4.4
1000	-5.2	-14.3	-14.3	-14.3	-14.4

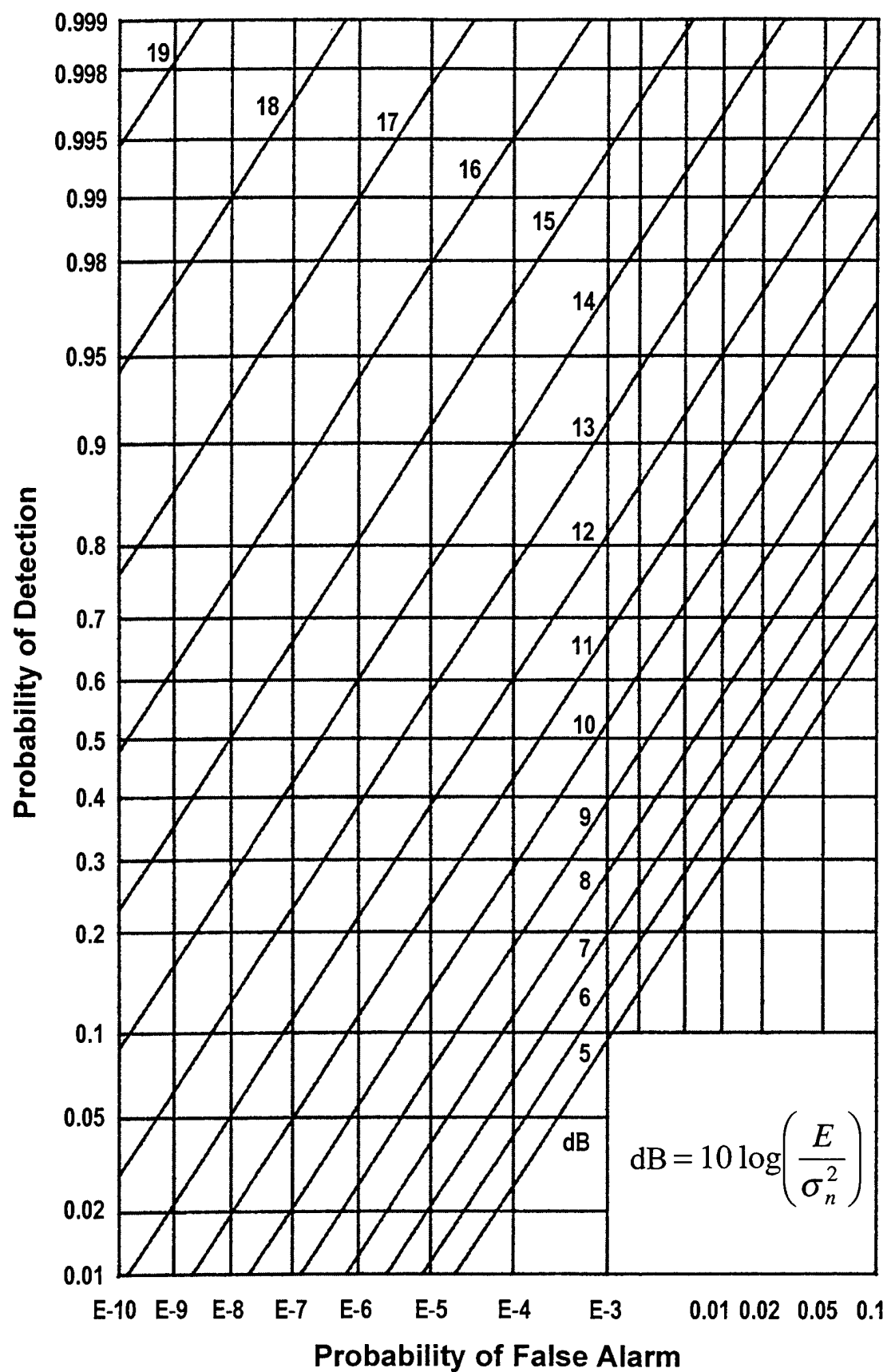


Figure 1. ROCs for Deterministic Signal

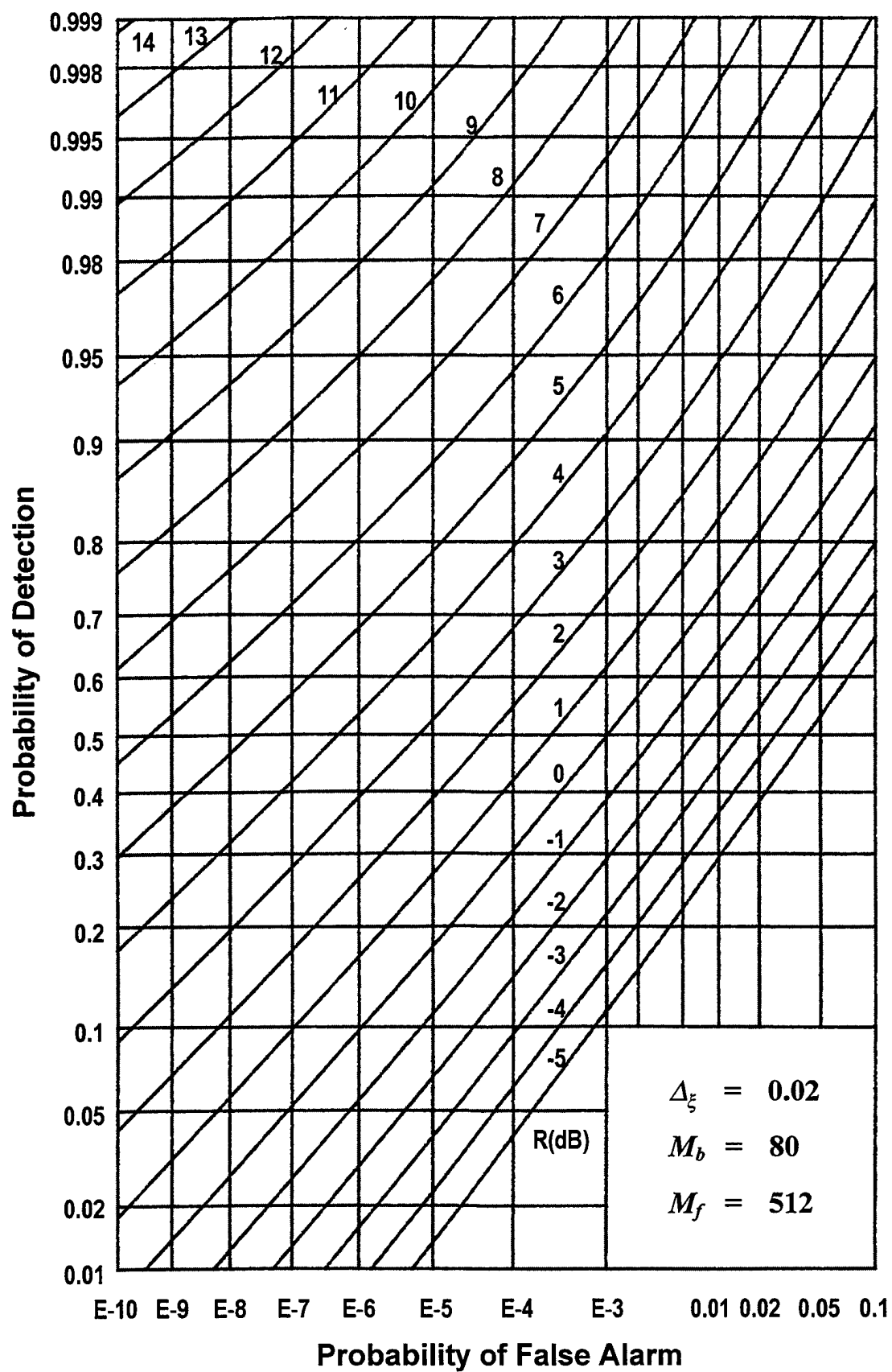


Figure 2. ROCs for Optimum Processor, $M = 10$

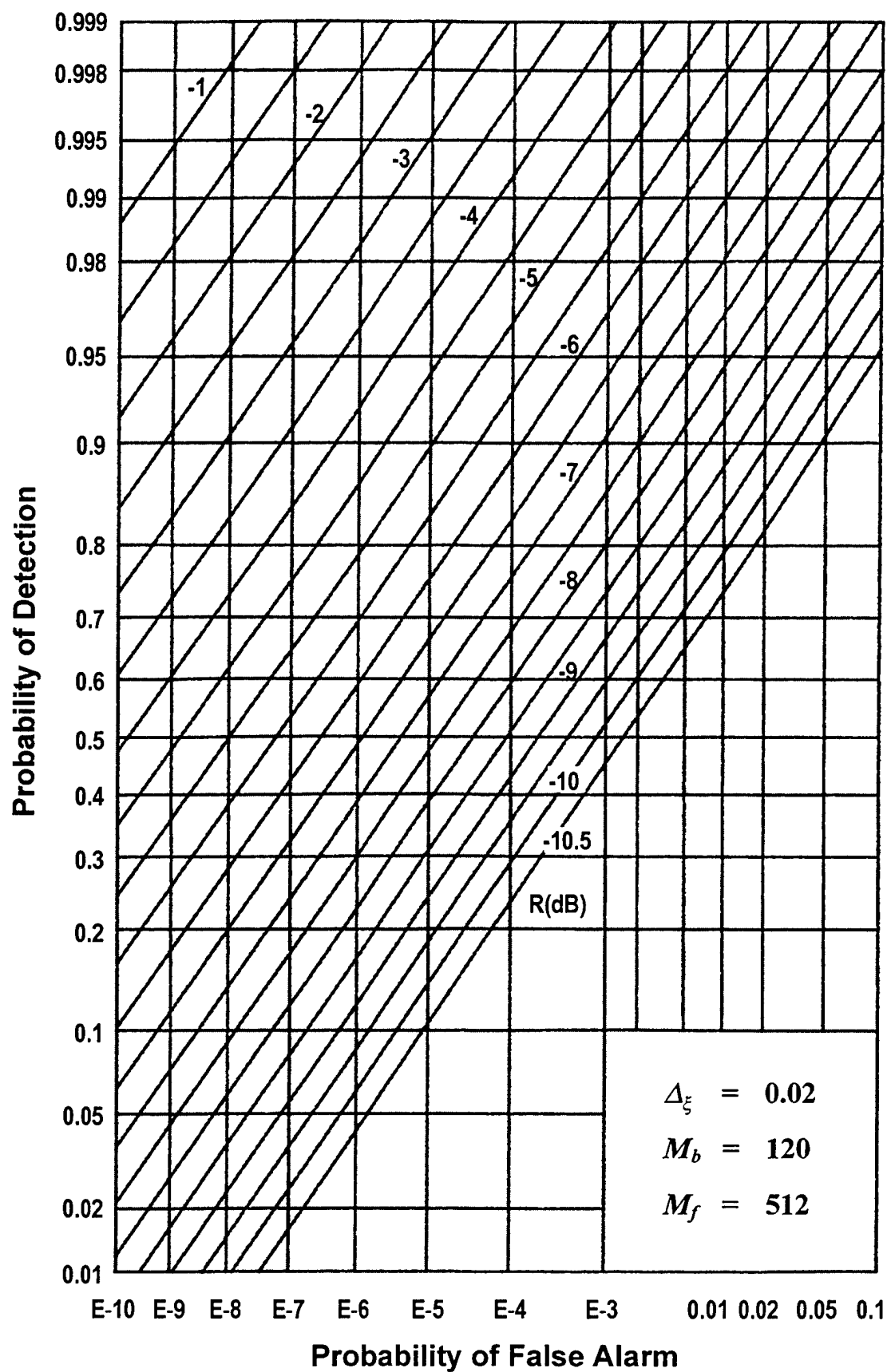


Figure 3. ROCs for Optimum Processor, $M = 100$

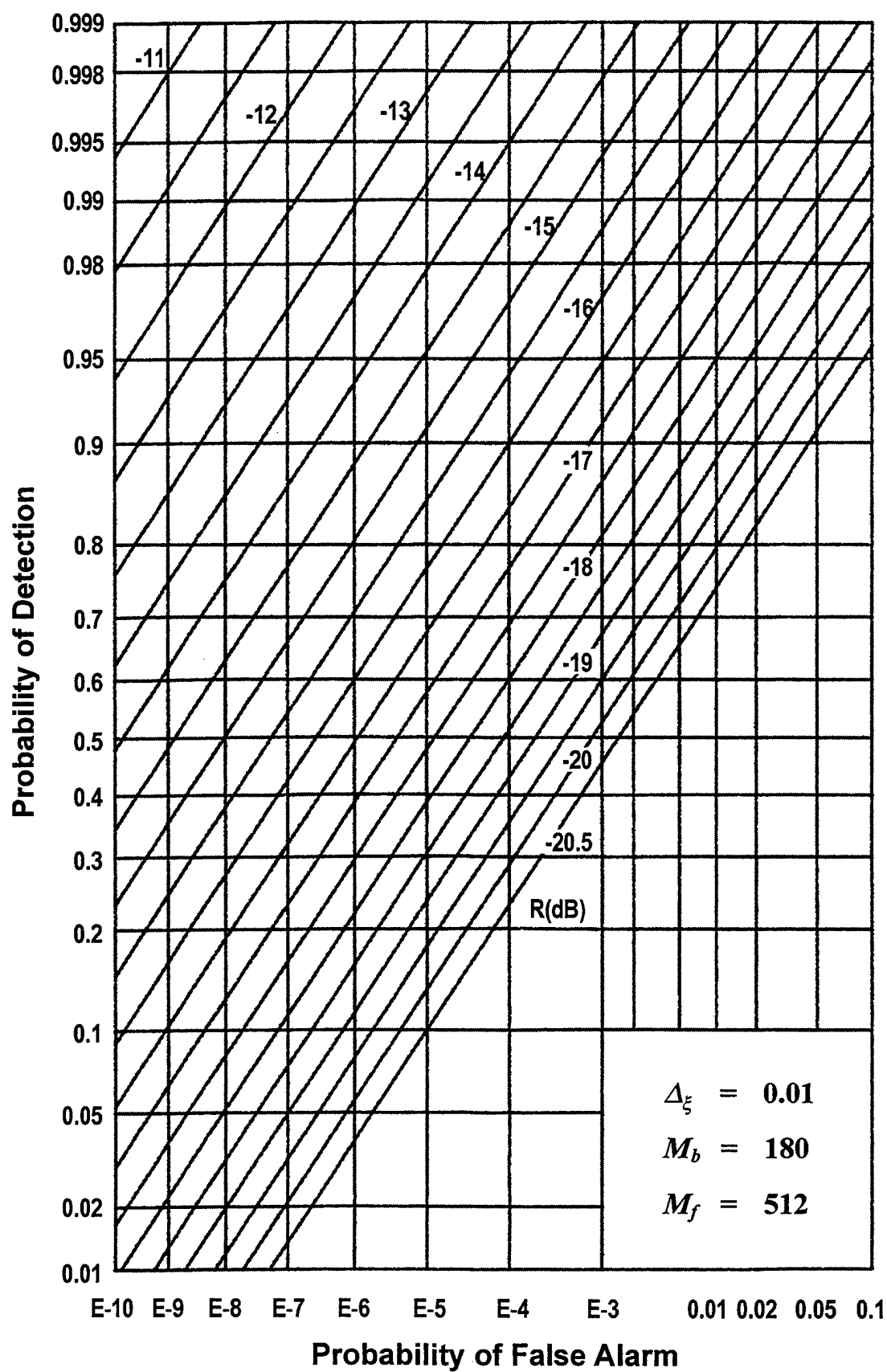


Figure 4. ROCs for Optimum Processor, $M = 1000$

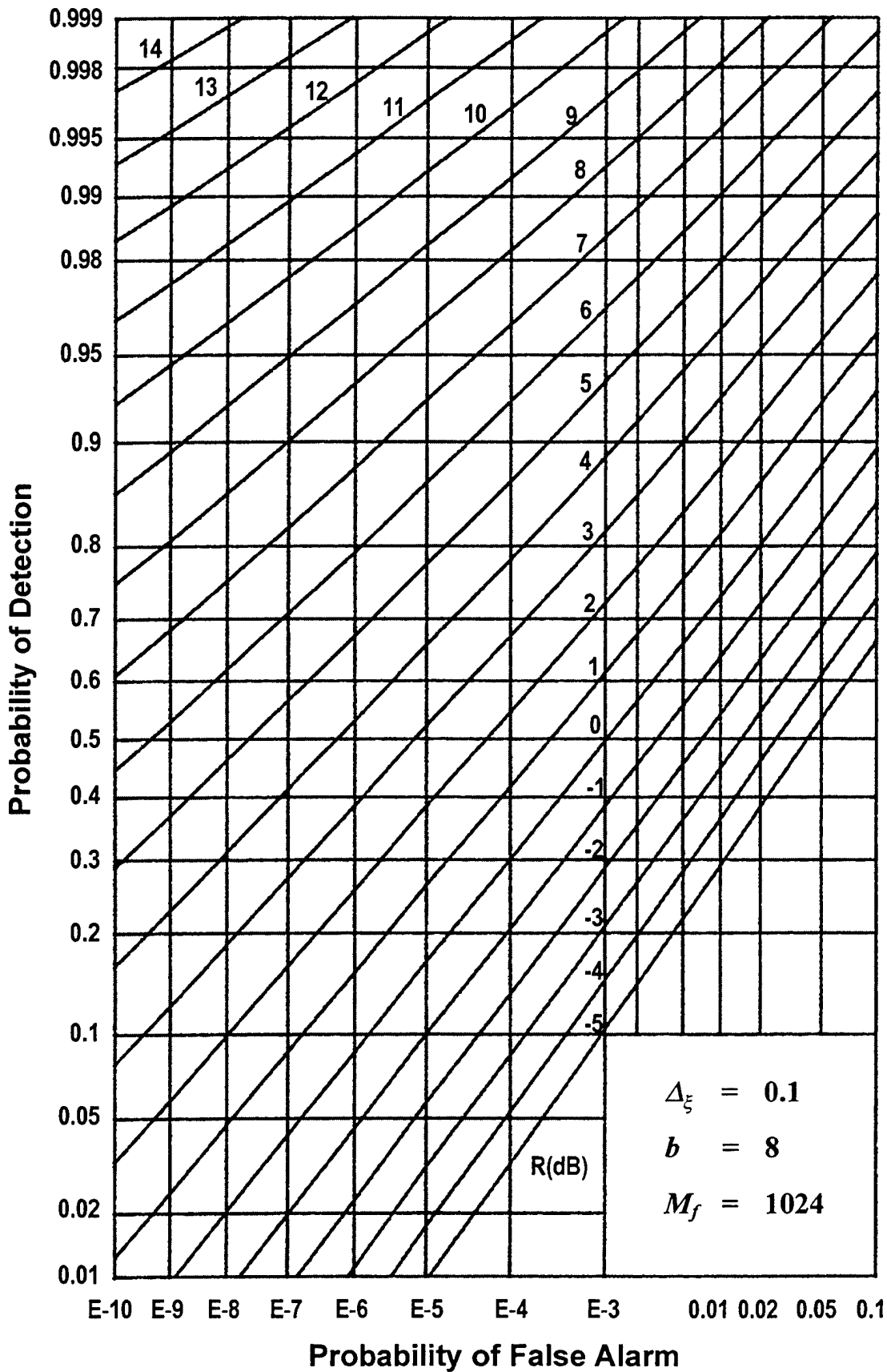


Figure 5. ROCs for Normalized Crosscorrelator, $M = 10$

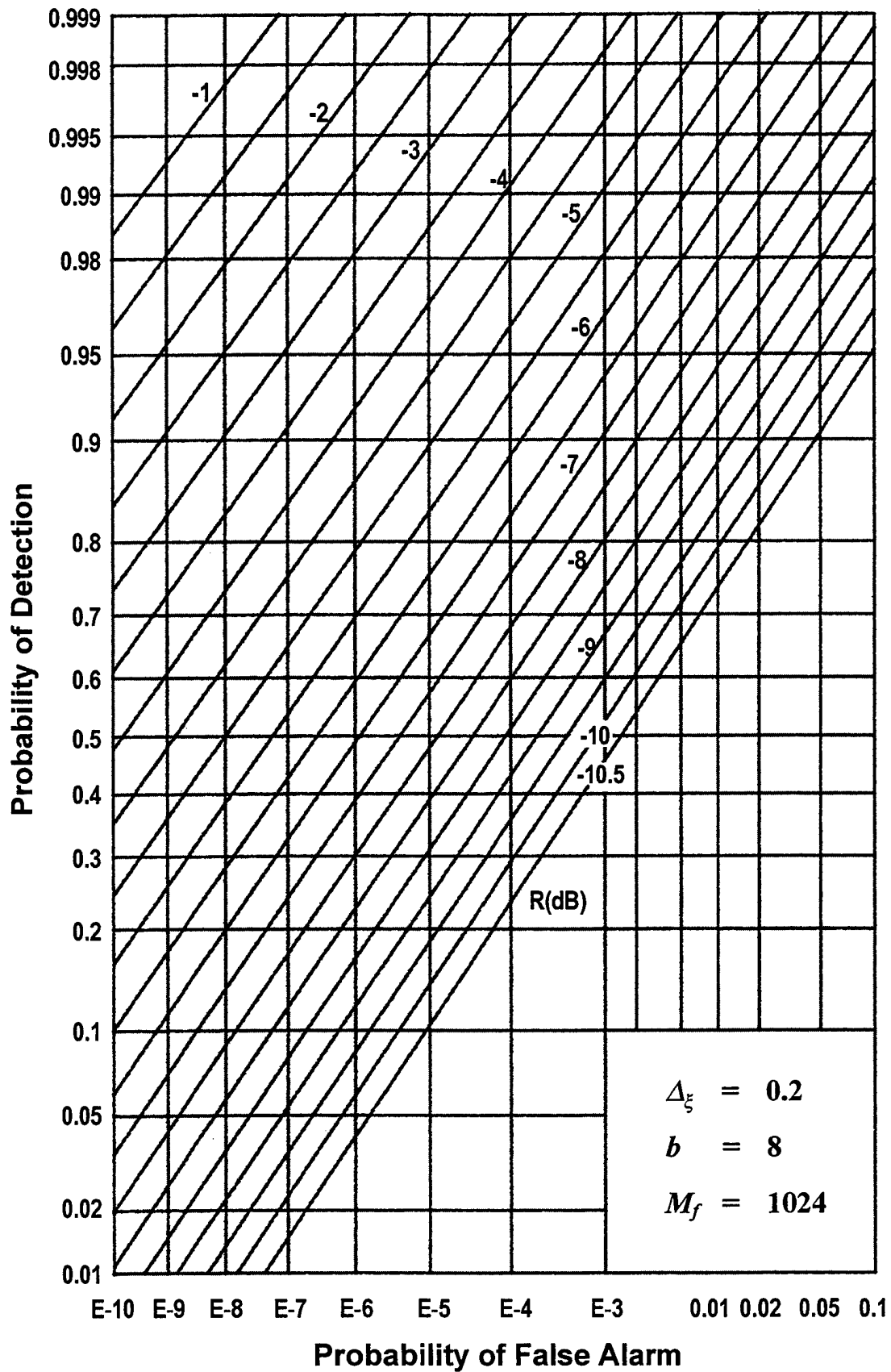


Figure 6. ROCs for Normalized Crosscorrelator, $M = 100$

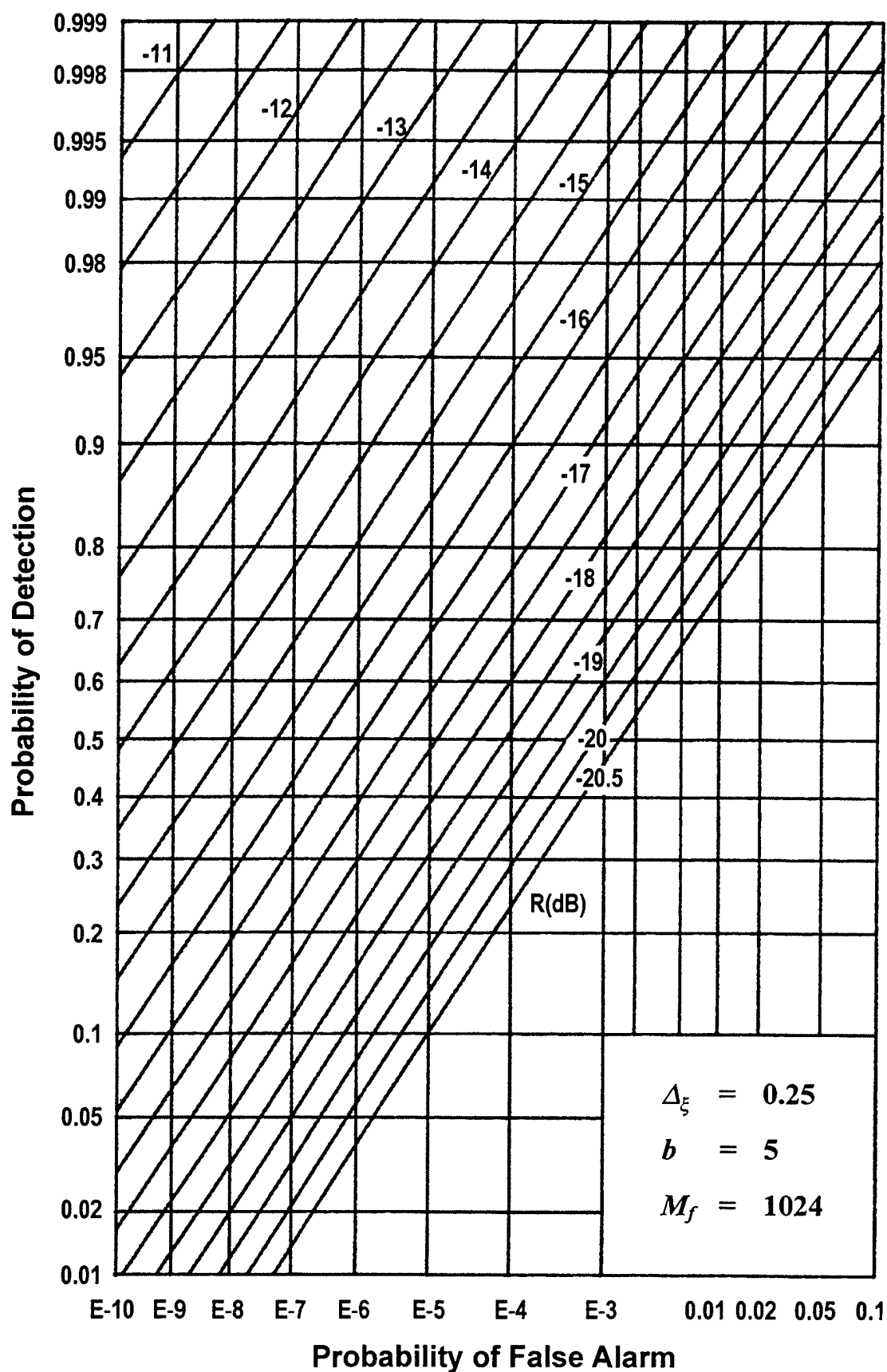


Figure 7. ROCs for Normalized Crosscorrelator, $M = 1000$

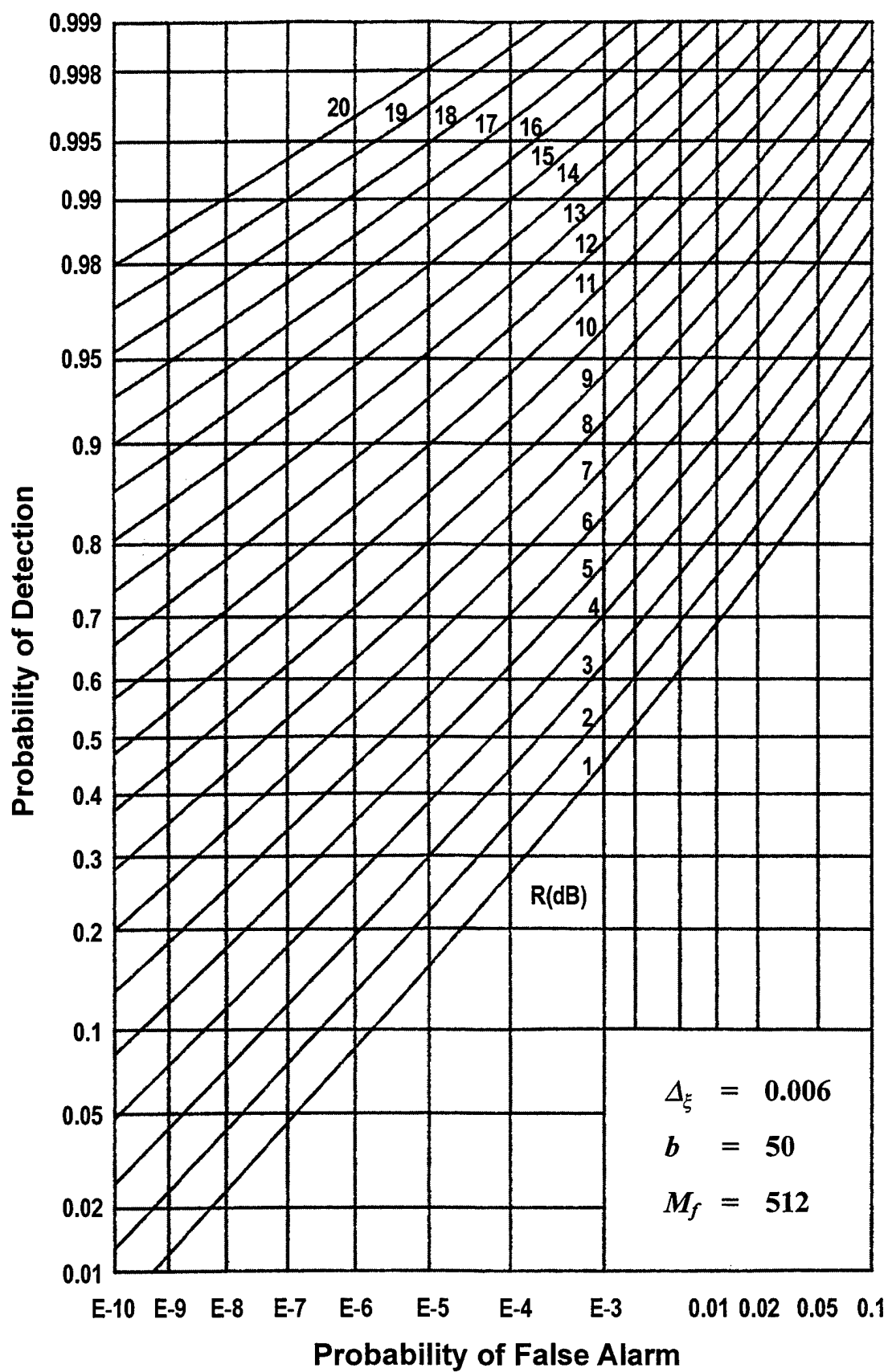


Figure 8. ROCs for Crosscorrelator, $M = 10$

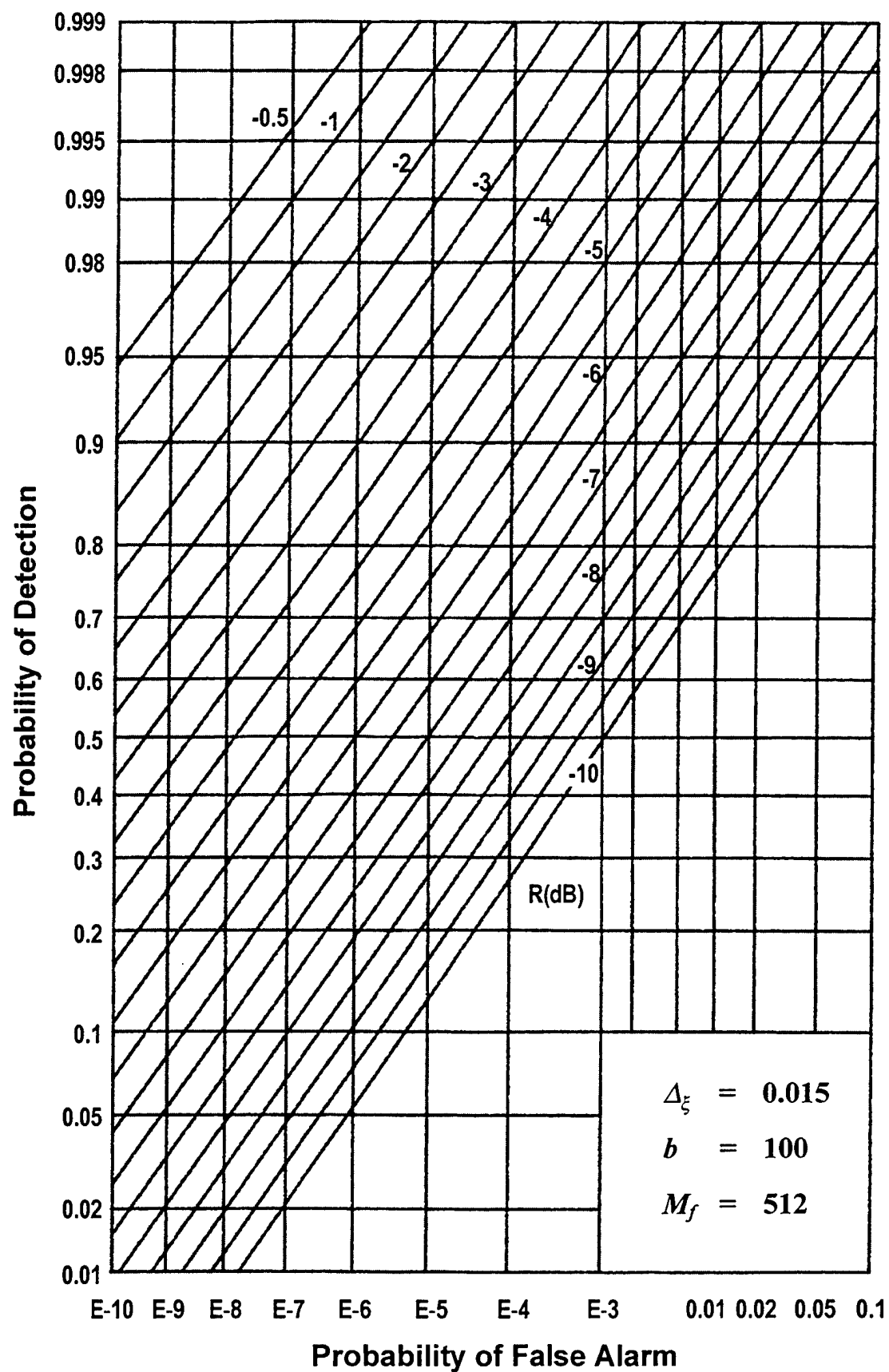


Figure 9. ROCs for Crosscorrelator, $M = 100$

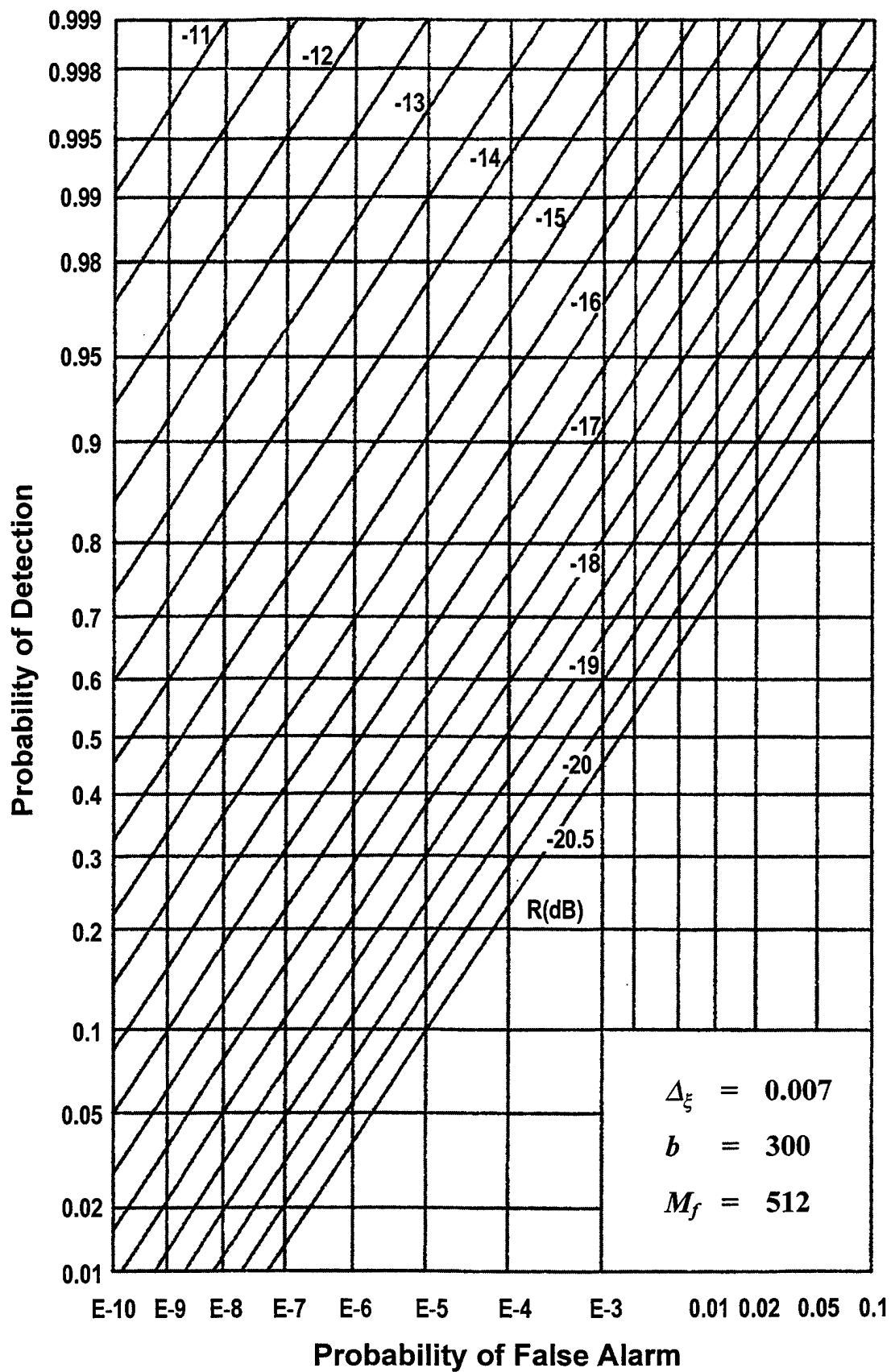


Figure 10. ROCs for Crosscorrelator, $M = 1000$

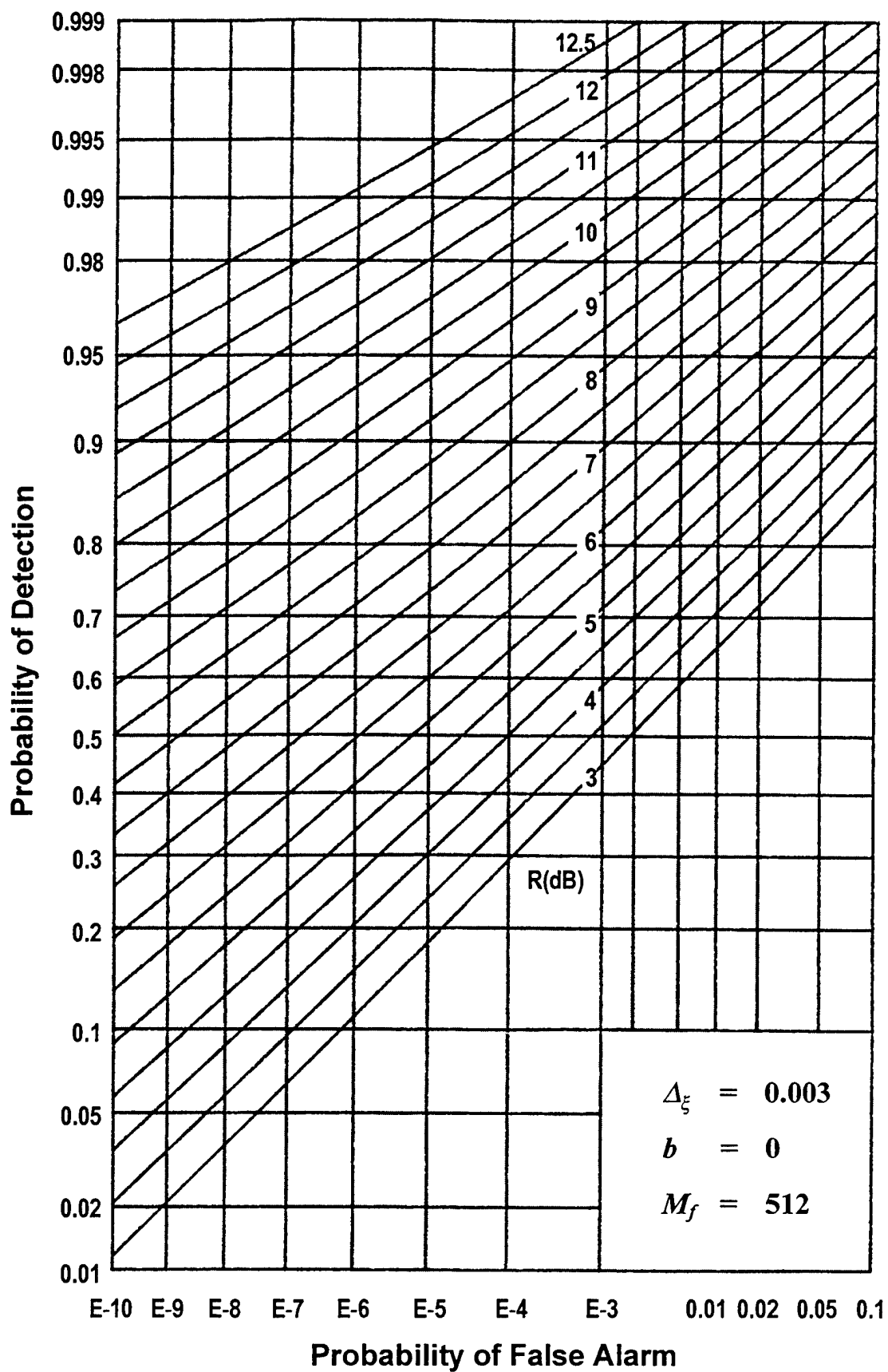


Figure 11. ROCs for Energy Detector, $M = 10$

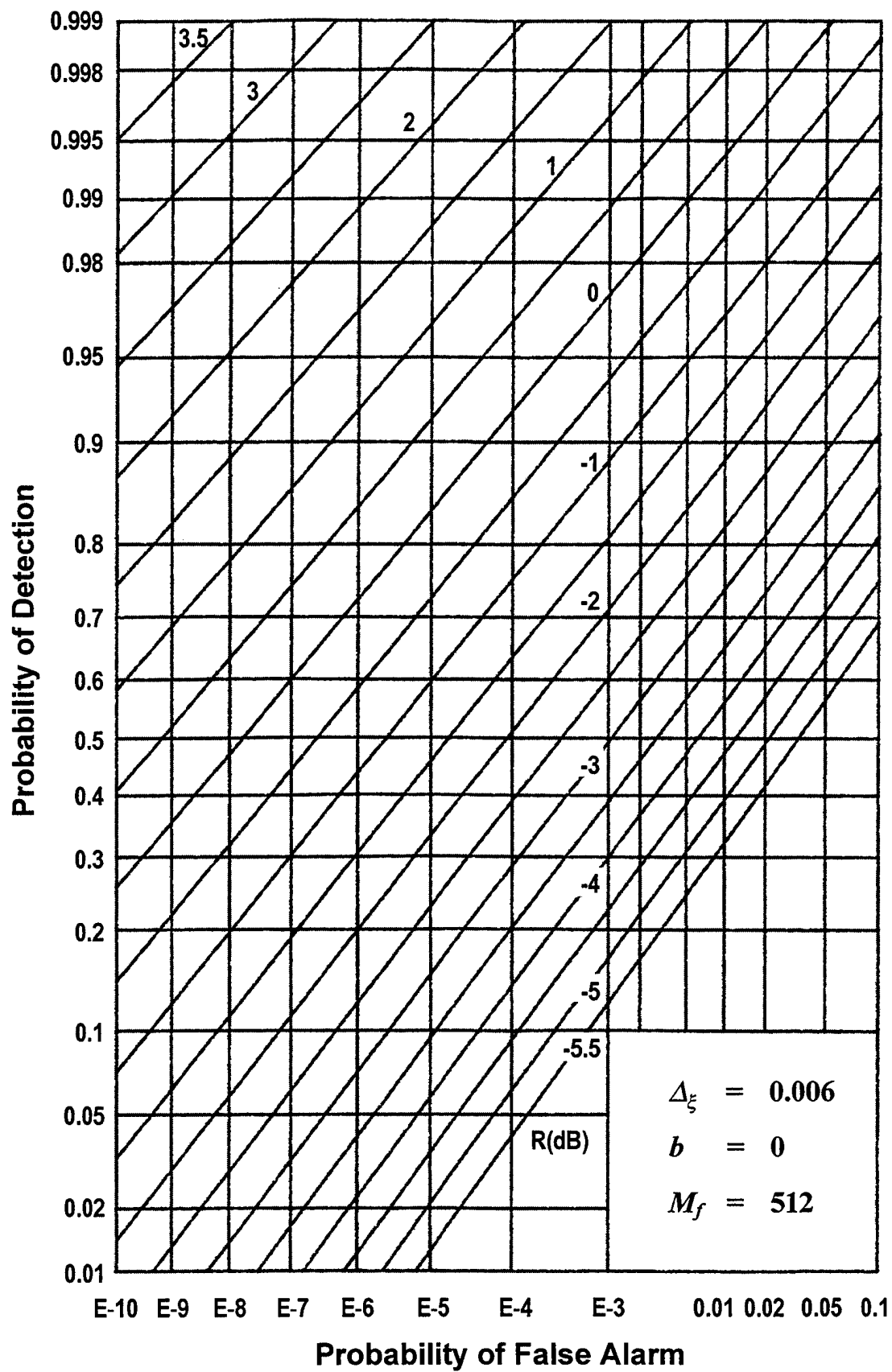


Figure 12. ROCs for Energy Detector, $M = 100$

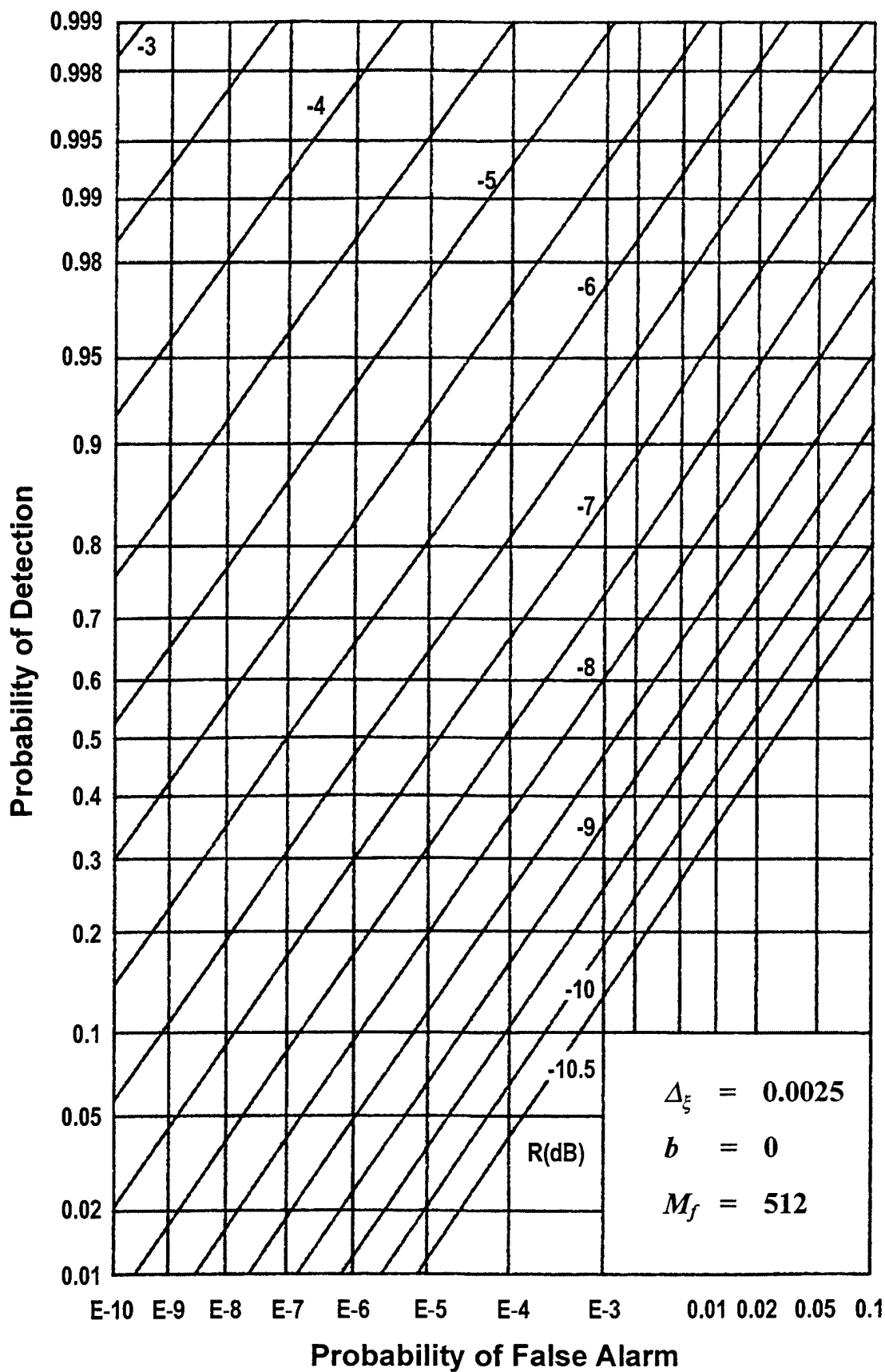


Figure 13. ROCs for Energy Detector, $M = 1000$

SUMMARY

The characteristic functions of the decision variables for the four different processors used to detect random signal transmissions have been derived in closed form and then numerically exercised to yield the corresponding accurate receiver operating characteristics. The number of data samples M available during the observation interval is arbitrary, as is the input signal-to-noise ratio.

The usual energy detector and crosscorrelator have operating characteristics that cross each other, the exact locations depending on the value of M and the particular signal-to-noise ratio. Neither of these processors performs as well as the normalized crosscorrelator, which itself performs within a small fraction of a decibel of the optimum processor. The obvious conclusion is to recommend use of the normalized crosscorrelator as the preferred detector, at least under the conditions utilized in this study.

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